Teorija brojeva

1) Dokazati

a) broj je deljiv sa 2 akko mu poslednja cifra deljiva sa 2.
b) ——— || ——— 3 akko mu zbir cifara deljiv sa 3.
c) ——— || ——— 4 akko mu dvocifreni završetak deljiv sa 4.
d) ——— || ——— 5 akko se završava sa 5 ili 0.
e) ——— || ——— 8 akko mu trocifreni završetak deljiv sa 8.
f) ——— || ——— 9 akko mu zbir cifara deljiv sa 9.
g) ——— || ——— 11 akko mu je razlika zbira cifara na parnim i neparnim mestima deljiva sa 11.

$$n = \overline{a_k a_{k-1} \ldots a_1 a_0} = a_k \cdot 10^k + a_{k-1} \cdot 10^{k-1} + \ldots + a_1 \cdot 10^1 + a_0$$

a)
$$a_0 + \underbrace{a_1 \cdot 10}_{} + \underbrace{a_2 \cdot 100}_{} + \underbrace{a_3 \cdot 1000}_{} + \ldots + \underbrace{a_{k-1} \cdot 10^{k-1}}_{} + \underbrace{a_k \cdot 10^k}_{}$$
$$\equiv_2 a_0 + 0 + 0 + 0 + \ldots + 0 + 0 = a_0$$

b)
$$a_0 + \underbrace{a_1 \cdot 10}_{} + \underbrace{a_2 \cdot 100}_{} + \underbrace{a_3 \cdot 1000}_{} + \ldots + \underbrace{a_{k-1} \cdot 10^{k-1}}_{} + a_k \cdot 10^k$$
$$\equiv_3 a_0 + a_1 \cdot 1 + a_2 \cdot 1 + a_3 \cdot 1 + \ldots + a_{k-1} \cdot 1 + a_k \cdot 1 = \sum_{i=0}^{k} a_i$$

c)
$$a_0 + \underbrace{a_1 \cdot 10}_{} + \underbrace{a_2 \cdot 100}_{} + \underbrace{a_3 \cdot 1000}_{} + \ldots + \underbrace{a_{k-1} \cdot 10^{k-1}}_{} + \underbrace{a_k \cdot 10^k}_{}$$
$$\equiv_4 \underbrace{a_0 + a_1 \cdot 10}_{} + 0 + \ldots + 0 + \ldots + 0 + 0 = 10a_1 + a_0 = \overline{a_1 a_0}$$

g)
$$a_0 + \underbrace{a_1 \cdot 10}_{} + \underbrace{a_2 \cdot 100}_{} + \underbrace{a_3 \cdot 1000}_{} + \underbrace{a_4 \cdot 10000}_{} + \ldots$$
$$\equiv_{11} a_0 - a_1 + a_2 - a_3 + a_4 - \ldots$$

$\boxed{1001 = 7 \cdot 11 \cdot 13}$

$1 \equiv_{11} 1$
$10 \equiv_{11} -1$
$100 \equiv_{11} 1$
$1000 \equiv_{11} -1$
$10000 \equiv_{11} 1$

2. Ispitati koji brojevi od 2 do 11, osim 7, dele broj 6477 0325 196.

2: ✓

3: $\cancel{6+4} + \cancel{7+7} + 0 + \cancel{3+2+5} + \cancel{1+9} + 6 = 10 + 14 + 10 + 10 + 6 = 50$
$3 \nmid 50 \Rightarrow$ ✗

4: $4 \mid 96$ ✓

5: ✗

6: treba 2 ✓, 3 ✗ $\Rightarrow$ ✗

8: $8 \mid 196$ ✗

9: $9 \nmid 50$ ✗

11: $6 - 4 + \cancel{7} - \cancel{7} + 0 \underbrace{- 3 + 2 - 5}_{} \underbrace{+ 1 - 9 + 6}_{} = 2 - 6 - 2 = -6 \quad 11 \nmid -6$ ✗

3. Naći ostatak pri deljenju broja 222^{555} brojem 11.

$222 \equiv_{11} 2 \Rightarrow 222^{\text{bilo šta}} \equiv_{11} 2^{\text{bilo šta}}$

$2^1 = 2 \equiv_{11} 2$
$2^2 = 4 \equiv_{11} 4$
$2^3 = 8 \equiv_{11} 8$
$2^4 = 16 \equiv_{11} 5$
$\boxed{2^5 = 32 \equiv_{11} -1}$
$2^6 = 64 \equiv_{11} 9$
$2^7 \equiv_{11} 18 \equiv_{11} 7$
$2^8 \equiv_{11} 14 \equiv_{11} 3$
$2^9 \equiv_{11} 6 \equiv_{11} 6$
$2^{10} \equiv_{11} 12 \equiv_{11} 1$

$$2^{555} = (2^5)^{111} \equiv_{11} (-1)^{111} = -1$$
$$222^{555} \equiv_{11} -1$$

Traženi ostatak je, dakle, 10.

4. Dokazati da je broj $3^{105} + 4^{105}$ deljiv sa 13.

$3^1 = 3 \equiv_{13} 3$
$3^2 = 9 \equiv_{13} 9$
$3^3 = 27 \equiv_{13} 1$

$$3^{105} = (3^3)^{35} \equiv_{13} 1^{35} = 1$$

$4^1 = 4 \equiv_{13} 4$
$4^2 = 16 \equiv_{13} 3$
$4^3 = 64 \equiv_{13} -1$

$$4^{105} = (4^3)^{35} \equiv_{13} (-1)^{35} = -1$$

$$3^{105} + 4^{105} \equiv_{13} +1 - 1 = 0 \Rightarrow 13 \mid 3^{105} + 4^{105}.$$

5. Naći ostatak pri deljenju broja 536^{324} brojem 7.

$536 \equiv_7 4 \qquad 536^{324} \equiv_7 4^{324}$?

$4^1 = 4 \equiv_7 4$
$4^2 = 16 \equiv_7 2$
$4^3 = 64 \equiv_7 1$

$$4^{324} = (4^3)^{108} \equiv_7 1^{108} = 1$$

Ostatak je 1.

6. Naći NZD (1996, 2006) koristeći Euklidov algoritam.

$2006 = \underline{1996} \cdot 1 + \underline{10}$
$1996 = \underline{10} \cdot 199 + \underline{6}$
$10 = \underline{6} \cdot 1 + \underline{4}$
$6 = \underline{4} \cdot 1 + 2$
$4 = 2 \cdot 2 + 0$

Poslednji nenula ostatak je NZD

$\Rightarrow NZD(1996, 2006) = 2.$

7. Ako je $NZD(a, b) = 1$ onda postoje $x, y \in \mathbb{Z}$ takvi da je $ax + by = 1$. Dokazati.

Posmotrojmo skup: $\{a, 2a, 3a, \ldots, b \cdot a\}$

Tvrdimo da ostaci po modulu b su različiti.

Pp. s. $\exists i, j$, $\boxed{1 \le i < j \le b}$ t.d. $i \cdot a \equiv_b j \cdot a$

$\Rightarrow b \mid ja - ia$
$\Rightarrow b \mid (j - i) \cdot a$
$\xrightarrow{NZD(a,b)=1} \Rightarrow b \mid (j - i)$ ↯ $\boxed{1 \le j - i < b}$

Ovo znači $\{a, 2a, 3a, \ldots, b \cdot a\}_{\equiv_b} = \{0, 1, 2, \ldots, b-1\}$

$\Rightarrow \exists\, x \in \{1, 2, \ldots, b\}$ t.d. $x \cdot a \equiv_b 1$

$\Rightarrow b \mid 1 - xa$
$\overset{\exists q}{\Rightarrow} b \cdot q = 1 - xa$
$\Rightarrow a \cdot \underbrace{\boxed{x}}_{x} + b \cdot \underbrace{\boxed{q}}_{y} = 1$

8. Ako $NZD(a, b) = d$, tada LINEARNA DIOFANTOVA JEDNAČINA $ax + by = c$ ima rešenje u skupu $\mathbb{Z}$ akko $d \mid c$. Dok.

($\Rightarrow$) Neka je $(x_0, y_0) \in \mathbb{Z}^2$ rešenje Dif. j-ne.

tj $\underbrace{a x_0}_{d \mid} + \underbrace{b y_0}_{d \mid} = c \Rightarrow d \mid c$

($\Leftarrow$) Pp. $d \mid c$. $\qquad ax + by = c \quad /: d$

$\underline{c = d \cdot k} \qquad \frac{a}{d} \cdot x + \frac{b}{d} \cdot y = k$

$NZD\left(\frac{a}{d}, \frac{b}{d}\right) = 1 \overset{Z.7.}{\Rightarrow} \exists$ celi brojevi x i y t.d.

celi brojevi

$$\frac{a}{d} x + \frac{b}{d} y = 1 \quad / \cdot k$$

$$\frac{a}{d} x k + \frac{b}{d} y k = k \quad / \cdot d$$

$$a \overset{x}{(xk)} + b \overset{y}{(yk)} = c$$

zaista postoji rešenje.

9. Ako je (x_0, y_0) jedno rešenje linearne Diofantove jednačine $ax + by = c$ (gde $NZD(a, b) = 1$) onda se sva rešenja mogu dobiti pomoću formula

$$x = x_0 + bt \qquad y = y_0 - at \qquad t \in \mathbb{Z}$$

Neka je (x_0, y_0) neko rešenje

$$\left.\begin{aligned} a x_0 + b y_0 &= c \\ a x + b y &= c \end{aligned}\right\} \; a(x - x_0) + b(y - y_0) = 0.$$

$$a(x - x_0) = b\underbrace{(y_0 - y)}_{at} \Rightarrow a \mid (y_0 - y) \Rightarrow \exists t \in \mathbb{Z}$$

$a \cdot t = y_0 - y$

$\cancel{a}(x - x_0) = b \cdot \cancel{a} \cdot t$

$x - x_0 = bt$

$$\boxed{\begin{aligned} x &= x_0 + bt \\ y &= y_0 - at \end{aligned} \quad t \in \mathbb{Z}}$$

10. Rešiti sledeće linearne Diofantove jednačine:

a) $5x + 7y = 1$

$NZD(5, 7)$

$7 = \underline{5} \cdot 1 + \underline{2}$
$5 = \underline{2} \cdot 2 + \underline{1} \quad \rightarrow \quad 1 = 5 - 2 \cdot 2 = 5 - 2 \cdot (7 - 5 \cdot 1) = \boxed{-2 \cdot 7 + 3 \cdot 5}$
$2 = 1 \cdot 2 + 0$

$NZD(5, 7) = 1$

$5 \cdot 3 + 7(-2) = 1$

$x_0 = 3, \quad y_0 = -2$

Sva rešenja: $\{(3 + 7t, -2 - 5t) : t \in \mathbb{Z}\}$

b) $7x + 29y = 4$

$29 = 7 \cdot 4 + 1$
$4 = 1 \cdot 4 + 0$

$1 = 29 - 7 \cdot 4$

$7(-4) + 29 \cdot 1 = 1 \quad / \cdot 4$
$7(-16) + 29 \cdot 4 = 4$

$(x_0, y_0) = (-16, 4)$

Skup svih rešenja: $\{(-16 + 29t, 4 - 7t) : t \in \mathbb{Z}\}$.

c) $3004x + 2014y = 7$

$3004 = 2014 \cdot 1 + 990$
$2014 = 990 \cdot 2 + 34$
$990 = 34 \cdot 29 + 4$
$34 = 4 \cdot 8 + \boxed{2}$ ← poslednji nenula ostatak
$4 = 2 \cdot 2 + 0$

$NZD(3004, 2014) = 2$

ali $2 \nmid 7$

jednačina nema rešenje.